A simple optical probe of transient heat conduction

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We use a laser beam and a stopwatch to investigate transient heat conduction in Plexiglas and glycerol samples chilled by ice water. The deflection of the laser beam is proportional to the thermal gradient in the sample. Measurements of the beam deflection allow us to calculate the thermal gradient as a function of time. Our empirical results fit the theoretical predictions very well and show an initial increase in the thermal gradient followed by a gradual decrease as the entire sample approaches the temperature of ice water. The procedure is simple and can be used as a lecture demonstration, an afternoon's experiment, or an extended investigation in an advanced laboratory course. © *2010 American Association of Physics Teachers.*
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I. INTRODUCTION

The theoretical treatment of transient heat conduction is a wonderful illustration of the use of separation of variables, Fourier series, and other advanced techniques. The concepts are simple and concrete, allowing students to appreciate the application of mathematics to everyday phenomena. As the mathematics increases in complexity, the experimental confirmation becomes more gratifying. A typical experiment makes use of thermocouples, whose insertion into samples requires some machining.[1] We adopt a simpler procedure and use a laser beam to probe the thermal gradient at selected positions in a transparent sample.

The experimental arrangement is shown in Fig. 1. A laser beam strikes the sample a distance z from the bottom of the sample. We establish a thermal gradient by bringing ice water into contact with the bottom of the sample. The thermal gradient produces a refractive gradient because the refractive index depends on temperature.[2] For typical materials and temperature ranges, dn/dT is approximately constant, and a single value of dn/dT is usually reported for a particular material.[3,4] Because dn/dT is typically obtained from measurements using HeNe lasers,[3,4] we too use a HeNe laser since dn/dT can depend on wavelength.

It is well known that the deflection of the laser beam is proportional to the refractive gradient in the sample.[5] If the refractive gradient of the sample is dn/dz at height z, then the beam is deflected along the wall a distance

$$h = -(dn/dz)LR, \tag{1}$$

where L is the length of the sample and R is the distance from the sample to the wall. The minus sign appears because we define h as a downward deflection, whereas z increases upward. We substitute $dn/dz=(dn/dT)(\partial T/\partial z)$ into Eq. (1) and find

$$h = -LR(dn/dT)(\partial T/\partial z), \tag{2}$$

where T is the temperature. By using Eq. (2), measurements of h as a function of time can be compared with theory.

We need a theory for $T(z,t)$ in our sample. The governing equation is

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial z^2}, \tag{3}$$

where D is the thermal diffusivity. The initial condition is

$$T(z,0) = T_a, \tag{4}$$

where T_a is the ambient temperature, because the sample is initially in thermal equilibrium at ambient conditions. The boundary conditions are

$$T(0, t>0) = 0\ ^\circ\mathrm{C} \tag{5}$$

and

$$\partial T(H,t)/\partial z = 0, \tag{6}$$

where H is the height of the sample. In writing Eq. (5), we assumed highly effective heat transfer between the sample and the ice bath. (For heat to flow into the ice bath, the temperature of the bottom of the sample must be higher than the temperature of the ice bath, but we assume that this temperature difference is vanishingly small.) Conversely, Eq. (6) assumes negligible heat transfer between the sample and the air. These approximations assume that air is much less dense than water, and the air's thermal conductivity is consequently less than 5% of water's thermal conductivity.[6] The validity of these approximations will be tested experimentally.

To solve Eq. (3), we use separation of variables, $T(z,t)=Z(z)\tau(t)$, substitute this form into Eq. (3), and obtain $Z(d\tau/dt)=\tau D(d^2Z/dz^2)$ or $(d^2Z/dz^2)/Z=(d\tau/dt)/(D\tau)$. This relation holds for all z and all t, which is possible only if both sides are equal to the same constant, $-k^2$. (For convenience, we choose $-k^2$ instead of k.) We then have $d\tau/dt=-Dk^2\tau$ and $d^2Z/dz^2=-k^2Z$. From here, most advanced students will be able to derive the final result

$$T(z,t) = \sum_{p=0}^{\infty} \frac{2T_a}{\left(p+\frac{1}{2}\right)\pi}\sin\left[\left(p+\frac{1}{2}\right)\pi z/H\right] \times\exp\left[-\left(p+\frac{1}{2}\right)^2\pi^2 Dt/H^2\right]. \tag{7}$$

In Eq. (2), we need $\partial T/\partial z$,

$$\frac{\partial T}{\partial z} = \sum_{p=0}^{\infty} \frac{2T_a}{H}\cos\left[\left(p+\frac{1}{2}\right)\pi z/H\right] \times\exp\left[-\left(p+\frac{1}{2}\right)^2\pi^2 Dt/H^2\right]. \tag{8}$$

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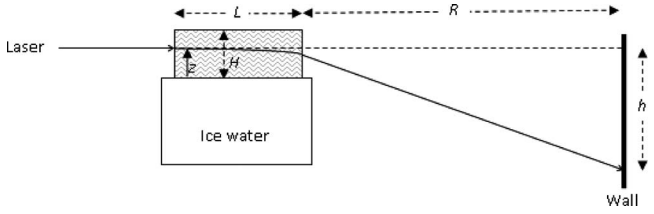

Fig. 1. The experimental arrangement. The laser beam is deflected by the thermal gradient in the sample.

II. PROCEDURE

A room-temperature rectangular sample of length L and height H is held in place by a clamp. A HeNe laser beam (with a wavelength of 632.8 nm) is directed lengthwise through the sample, perpendicular to the end surfaces, and at a distance z above the bottom surface. The beam travels to a wall a distance $R=9.76$ m, with R being much greater than L and H. After securing the sample and the laser, a bowl of ice water is raised from below until the water just contacts the bottom surface of the sample. We define this time of contact as $t=0$.

At regular time intervals the position of the center of the beam on the wall is marked in pencil. The vertical displacement of the beam from its initial position is measured and recorded. This procedure is repeated with a Plexiglas block ($H=7.88$ cm and $L=6.06$ cm) for three values of z and with a rectangular flask of glycerol ($H=2.38$ cm and $L=8.50$ cm) for two values of z. In all cases, the sample begins at room temperature. At least 1 day elapses between experiments so that the sample's initial condition is unaffected by previous experiments.

III. RESULTS AND DISCUSSION

Results for Plexiglas are shown in Fig. 2. We used D and dn/dT as fitting parameters. The fitting procedure is simple because dn/dT affects only the height (not the shape) of the curve. First D was adjusted until the theoretical peak occurred near the time of the measured peak deflection; then dn/dT was adjusted until the theoretical curve had nearly the same height as the experimental curve. Small adjustments were made to improve the fit visually. The method of least-squares would allow us to determine the parameters with greater precision. However, we know z to only two significant digits, so improved precision is not justified.

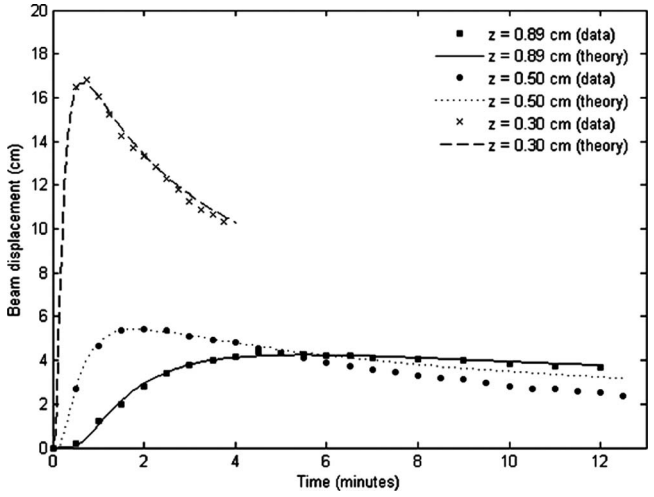

Fig. 2. Theoretical curves fitted to the measured beam displacements for a Plexiglas sample. For $z=0.89$ cm the fitting parameters are $D=1.2\times10^{-3}$ cm^2/s and $dn/dT=-5.6\times10^{-5}$ K^{-1}. For $z=0.50$ cm we obtained $D=1.1\times10^{-3}$ cm^2/s and $dn/dT=-4.0\times10^{-5}$ K^{-1}. For $z=0.30$ cm we found $D=1.1\times10^{-3}$ cm^2/s and $dn/dT=-7.4\times10^{-5}$ K^{-1}.

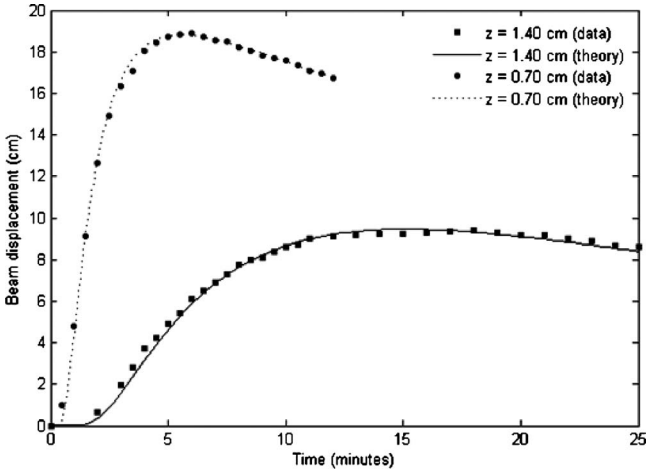

Fig. 3. Theoretical curves fitted to the measured beam displacements for a glycerol sample. For $z=1.40$ cm the fitting parameters are $D=8.3\times10^{-4}$ cm^2/s and $dn/dT=-1.49\times10^{-4}$ K^{-1}. For $z=0.70$ cm, the fitting parameters are $D=7.3\times10^{-4}$ cm^2/s and $dn/dT=-1.39\times10^{-4}$ K^{-1}.

For $z=0.89$ cm we found $D=1.2\times10^{-3}$ cm^2/s and $dn/dT=-5.6\times10^{-5}$ K^{-1}. For $z=0.50$ cm we used $D=1.1\times10^{-3}$ cm^2/s and $dn/dT=-4.0\times10^{-5}$ K^{-1}. For $z=0.30$ cm we obtained $D=1.1\times10^{-3}$ cm^2/s and $dn/dT=-7.4\times10^{-5}$ K^{-1}. Our results for D are within 10% of one another and the value 1.06×10^{-3} cm^2/s found in Ref. 7. However, our results for dn/dT differ significantly from one another and from the reported value[3] of -1.2×10^{-4} K^{-1}. We believe that the error in our results is due to the difficulty of getting the ice water to wet the entire bottom surface of the sample without submerging some of the sample. If the ice water rises even 1 mm over the bottom of the sample, heat can flow through the sides of the sample into the ice water, compromising the validity of our one-dimensional model. Despite this difficulty, we find good qualitative agreement between the theoretical curves and the measured data.

Results for glycerol are shown in Fig. 3. For $z=1.40$ cm the best fit is achieved with $D=8.3\times10^{-4}$ cm^2/s and $dn/dT=-1.49\times10^{-4}$ K^{-1}. For $z=0.70$ cm we find $D=7.3\times10^{-4}$ cm^2/s and $dn/dT=-1.39\times10^{-4}$ K^{-1}. Qualitatively, the fit appears excellent. Quantitatively, there is modest agreement between our results and the published values for glycerol, $D=9.4\times10^{-4}$ cm^2/s and $dn/dT=-2.4\times10^{-4}$ K^{-1}.[4]

Because our measurements are proportional to $\partial T/\partial z$ rather than T, we can directly observe that $\partial T/\partial z$ increases to a maximum value before decreasing. The explanation is as follows. The thermal gradient is initially zero when the sample is in equilibrium with the ambient conditions. Once contact is made with the ice water, a large thermal gradient appears at the very bottom of the sample. As heat consequently flows downward, the thermal gradient spreads upward through the sample. When the "cold front" sweeps

through a layer, the thermal gradient there has its greatest value because the lower layer is cold, while the upper layer is still warm. After the cold front has passed through, the upper layer has also begun to cool, and the thermal gradient decreases. The cold front passes lower positions at earlier times, as illustrated by our data: For small z, the maximum displacement occurs earlier. It is instructive for students to explain these qualitative features of the data. These observations of $\partial T/\partial z$ are more interesting than observations of T, which monotonically decreases everywhere in the sample.

IV. CONCLUSIONS

Within minutes, the experiment demonstrates that the thermal gradient of a sample dipped in ice water increases and then decreases. This phenomenon is an accessible mystery for students at all levels. Introductory students can understand the explanation qualitatively, and more advanced students can supplement their understanding with the full mathematical treatment. The experiment can be performed as a lecture demonstration, an afternoon's laboratory activity, or an extended investigation of the thermal and optical properties of various transparent materials.

ACKNOWLEDGMENT

The authors gratefully acknowledge Phil Segre for suggesting the experiment.

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Science and Complexity

Warren Weaver published a seminal paper in 1948 in Science that discussed science and complexity. Two-body problems, the main focus of classical physics, are described as simple. The general behavior of millions of reasonably identical bodies, each acting unpredictably, can be calculated with the aid of probability theory and statistical mechanics. These he called the problems of disorganized complexity. He then turned his attention to systems involving numbers of particles, agents, or bodies, more than two or three but much less than millions. Systems of this kind are encountered in biological, medical, psychological, economic, social, and political sciences. They behave in an organized way. He called these the problems of organized complexity and noted that, although they are well described, the underlying processes are poorly understood. Weaver then asserted that, within the next fifty years, science must learn to deal with the problems of organized complexity.

George A. Cowan, *Manhattan Project to the Santa Fe Institute* (University of New Mexico Press, 2010) p. 144.