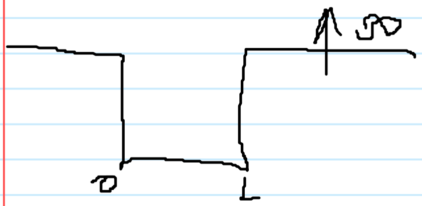
Sloshing States:
Recall the distinction between $\psi(x)$ and $\Psi(x,t)$.
Little psi, and big psi.

For the case of the infinite square well---we will take a linear combination of stationary states Ψ_n and find that the probability distribution is now time dependent =sloshing state.



Consider normalized states for infinite square well. from 0 to L

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad E_n = \frac{n^2 h^2}{8mL^2}$$

$$\Psi_n(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-i\left(\frac{E_n}{\hbar}\right)t}$$

from 0 to L
= 0 outside

When we take a linear combination we may include any desired relative amount of state "1" and state "2" , and so on. But we must ensure that we maintain---RENORMALIZATION in the end.

Recall — norm is

$$\int_{-\infty}^{\infty} \Psi_n^* \Psi_n \, dx \Rightarrow \int_0^L \frac{2}{L} \sin^2(\ ) \, dx$$

Now what if we add several states

$$\Psi(x,t) = \sum_n A_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-i\omega_n t}$$

but I must renormalize &

Phase is usually ignored

The A_n are an entirely new set of constants.
I can choose how big A_1 is relative to A_2 or A_3
BUT--must renormalize

A_n complex

We write out the process considered for renormalizing

We take a look at the probability density for Ψ_{total} as a supeposition of states summed over n or n'

$$\Psi^* \Psi = \left(\sum_n A_n^* \psi_n e^{+i\omega_n t} \right) \left(\sum_{n'} A_{n'} \psi_{n'} e^{-i\omega_{n'} t} \right)$$

We may think of the individual terms (which will need to be integrated) as forming a matrix of n n' product terms

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & \cdots \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{matrix} & \begin{pmatrix} \quad & \quad & \quad & \quad \\ \quad & & & \\ \quad & & & \\ \quad & & & \end{pmatrix} \end{array}$$

$$\begin{aligned} n, n' = \;& 1,1 \\ & 1,2 \\ & 1,3 \\ & \vdots \\ & 2,1 \\ & 2,2 \\ & 2,3 \\ & \vdots \\ & \text{etc} \end{aligned}$$

$$= \sum_n A_n^* A_n \psi_n^* \psi_n * 1 + \sum_{\substack{n \\ n \neq n'}} \sum_{n'} \text{other terms}$$

For the diagonal terms where n=n' the time dependent term cancels out (stationary state terms). For the off diagonal---the time dependent terms remain (n not equal to n'). The off diagonal terms are time dependent probability distribution

HOWEVER---FOR JUST NORMALIZING PURPOSES We want to look at the integrals for the individual terms (so just throw in the integral over dx from x=0 to L)

Now discuss integrals $\int_{\text{all } x} \Psi^* \Psi \, dx$

We have terms like

$$\int_{\substack{\text{whole}\\ \#\text{ half}\\ \text{cycles}}} \left(\frac{2}{L}\right) \sin\left(\frac{n\pi x}{L}\right) * \sin\left(\frac{n'\pi x}{L}\right) dx$$

$$= \delta_{nn'} \rightarrow 0 \text{ when } n \neq n'$$

all off diagonal terms $\rightarrow 0$

1) orthogonal
2) projection operation
3) scalar product

$$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \quad \hat{n}' \bullet \hat{n} = \delta_{n'n}$$

Recall that we are looking the integral portion of each term. There is also the multiplier of $A_n^* A_{n'}$ to include in each integral term---but we are looking at the remaining parts because we know the states and form of integrals (A's may be decided upon later).

WHEN NORMAILIZING

diagonal terms leave us with

$$1 = \sum_n |A_n|^2 \rightarrow A_1^2 + A_2^2 + A_3^2 + \cdots = 1$$

the A_ns are chosen to give an initial state

Fourier $\rightarrow$ can start with any $\Psi(x,0)$ we want by adding different states together

A_ns renormalize.

OK--so for purposes of normalization, each stationary state acts just like a unit vector----

$$\vec{A}_{tot} = A_1 \hat{n}_1 + A_2 \hat{n}_2 + \cdots$$

However the underlying probability distribution (before integrating) remains time dependent---keep the cross terms. Example

Lets look at simple case $A_1 = A_2$

$$\Psi(x,0) = A\,\psi_1 + A\,\psi_2$$

$$A^2 + A^2 = 1 \longrightarrow A = \frac{1}{\sqrt{2}}$$

$$\Psi(x,t) = \sqrt{\frac{2}{L}}\frac{1}{\sqrt{2}}\left(\sin\left(\frac{\pi x}{L}\right)e^{-i\omega_1 t} + \sin\left(\frac{2\pi x}{L}\right)e^{-i\omega_2 t}\right)$$

$$\Psi^*\Psi = \frac{1}{L}\left(\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)e^{+i\omega_1 t - i\omega_2 t} + \sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)e^{i\omega_2 t - i\omega_1 t}\right)$$

The last 2 terms combine to give $e^{i(\omega_2-\omega_1)t} + e^{-i(\;)t}$
$$= 2\cos[(\omega_2-\omega_1)t]$$

$$\Psi^*\Psi = \frac{1}{L}\left[\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + \underbrace{2\cos[(\omega_2-\omega_1)t]}_{\text{time dep term}}\sin\frac{\pi x}{L}\sin\left(\frac{2\pi x}{L}\right)\right]$$

plot $\Psi^*\Psi(x,t)$ as animation

What does this mean. So, if I have a particle in the state given---then the probability distribution sloshes back and forth in the well. ---the overall normalization remains constant (I am not growing or losing particles).

For the state we chose above---there is an equal probability of finding the particle in state 1 (with energy E_1) or state 2 (with energy E_2

We might ask what the average energy is for that system.

$$E_{Ave} = \langle E \rangle = A_1^2 E_1 + A_2^2 E_2$$

$$E_n = \frac{n^2 h^2}{8 m L^2} \qquad \frac{1}{2} E_1 + \frac{1}{2}(4 E_1)$$

$$= 2.5 E_1$$

There is no single state with energy 2.5 E_1 but for this mix of states, this will be the average. Half the time I will measure the state of the system to be E_1 the other half I will measure E_2 (which is 4*E_1)

Just like a single "die" 1/6(1+2+3+4+5+6)=3.5---as the average roll value, yet a single roll will never yield "3.5".

Lets look at another state for this system:

Given

$$\Psi_A(x,0) = \sqrt{\frac{1}{6}}\,\psi_1 + \sqrt{\frac{1}{3}}\,\psi_2 + \sqrt{\frac{1}{2}}\,\psi_3$$

Where $\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n \pi x}{L}\right)$ $\qquad E_n = \frac{\hbar^2 k_n^2}{2m}$

$$k_n = \frac{n\pi}{L}$$

— write out $\Psi(x,t) \rightarrow$

$\Psi^* \Psi \rightarrow$ Longer

$\rightarrow$ 3 cross terms w/related freqs

- Expectation Value of Energy

$$\langle E\rangle = \overline{E} = \frac{1}{6}E_1 + \frac{1}{3}E_2 + \frac{1}{2}E_3$$

$$= \frac{1}{6}E_1 + \frac{4}{3}E_1 + \frac{9}{2}E_1$$

$$= 6E_1 \rightarrow \text{do it}$$

- In a single measure of Energy do you ever measure "$6E_1$"

- What energies are measured & with what probability

$E_1 \longrightarrow \frac{1}{6}$ of time

E_2 $\frac{1}{3}$ of time

E_3 $\frac{1}{2}$ of time

Adds to 1

We also may go back to the probability distribution as a function of time and note that we will have cross terms involving states 1,2 and 1,3, and 2,3 ---so there will be a more complicated sloshing (beating---in time it's called beats, in space it's called interference) of the probability distribution.

Can we construct a new wavefunction that is orthogonal to the original (the three state example). All we need to do is ensure that the dot product (scalar product, projection operation) yields zero---just like dotting two vectors.

Construct a new wavefunction that is orthogonal to the original Ψ_A but uses the same 3 states & has the same probabilities of each

— hint — phase

$$\Psi_B = \sqrt{\tfrac{1}{6}}\,\psi_1 + \sqrt{\tfrac{1}{3}}\,\psi_2 - \sqrt{\tfrac{1}{2}}\,\psi_3$$

orthog means

$$\int_0^L \Psi_B^* \, \Psi_A \, dx = 0$$

$$= \frac{1}{6} + \frac{1}{3} - \frac{1}{2} = 0 \quad \checkmark$$

Why — the ψ's formed an orthonormal basis set

— the momentum ~~of a stationary state~~ is $\hbar k_n \rightarrow p_n = \pm \hbar k_n$

— the average momentum is zero

A particle stuck in an infinite square well has an average momentum of zero---duh.   It does not go anywhere.

We can ask the more relevant question, what is the average magnitude of the momentum for one of my BIG psi states.

— What is the average of the magnitude of momentum

denoted $|p|$ or $\langle |p| \rangle$

$$\langle |p| \rangle = \frac{1}{6}\left(\hbar \frac{1\pi}{L}\right) + \frac{1}{3}\hbar\left(\frac{2\pi}{L}\right) + \frac{1}{2}\hbar\left(\frac{3\pi}{L}\right)$$

Probability of each $|\hbar K_n|$ occuring!

Takeaway concepts

- We have looked at superpositions of stationary states
  - The probability distribution of a single stationary state is indeed STATIONARY
- A superposition of states has a time dependent probability distribution
  - At one time it is likely to find particle at position range A
  - At another time less likely at A --more likely at B
  - OH---yeah, so the particle is moving---
- Orthogonality helps easily calculate the expectation value (average) of any quantity for known stationary state combinations.
  - Just do the averaging for the given coefficients of the day.
- Warning for the future---you and I may not even use the same orthonormal basis sets---we may not even use the same variables.