Ch. 6 Planck's constant---blackbody radiation

Some discussion:

You are hot, therefore you glow!   ---OK---better way to say---all objects with temperature (thermal energy) give off electromagnetic radiation.

We want to know the spectrum?---How hot, how much light of each color?

Some assumptions:

- Light (em radiation) is quantized---huh--what?
- It is lumpy--why lumpy--how big are the lumps?
- Objects with T emit radiation
- Classical predictions do not agree with observations
- The emitters involve electrons and nuclei---atoms.
  - These "Atoms" can be treated as "oscillators"
- Oscillators and a bit about quantization is most of the assumption.
- Also we assume "thermalization"
  - That means if energy goes into a system (any flavor---any frequency) ---the energy somehow gets scrambled and leaks out any way it can find---via the emitting oscillators
  - The oscillators take that energy (anything we throw at it) and spread the energy out statistically--and out it comes--thermalized.

Statistics and vibrations run the process.

Get yourself a good slinky---the metal toy---"A slinky, a slinky……."

Have someone hold one end (still). You gently shake the other.

As you play with your slinky you notice there are two ways that you can increase the energy of the wave on the spring.

1) You can shake the slinky with bigger amplitude
2) You can shake the slinky with higher frequency.

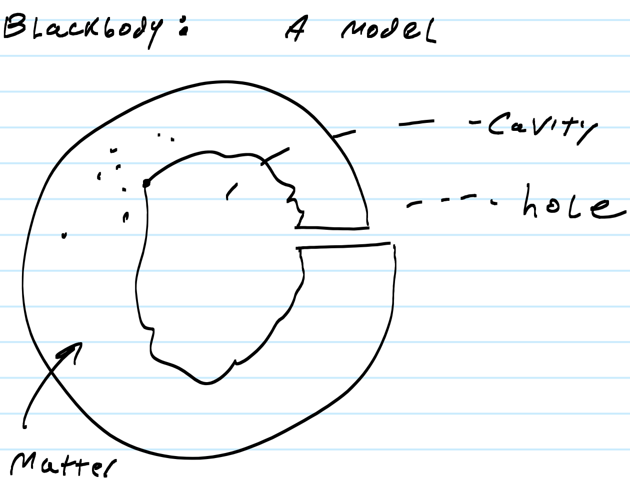
Either of these requires you to do more work. Either of these puts more energy into the standing wave of the system.

OH---and really important---the slinky can only maintain a vibration "good vibrations" if it is a standing wave mode of the system. QUANTIZED.

The classical assumption for the oscillating systems was that they could support any vibrational frequency (on each oscillator). The true nature is QUANTIZATION.

Wait--what is a blackbody emitter/radiator?
Energy goes in and becomes thermalized---it doesn't come out. At least it does not come out without undergoing the thermalization process--which is basically what we are modelling.

We will start with a lump of matter---you can consider it very cold so the particles making up the matter do not move. We let energy in. The energy is localized---but internal collisions and interactions tend to spread the energy out. ---2nd law of thermodynamics (energy spreads out).



Send LASER light into the hole, matter warms up---let the system reach equilibrium, and see what kind of light comes out (a whole spectrum--with relative intensities).

Before starting at some of the math---here are some results.

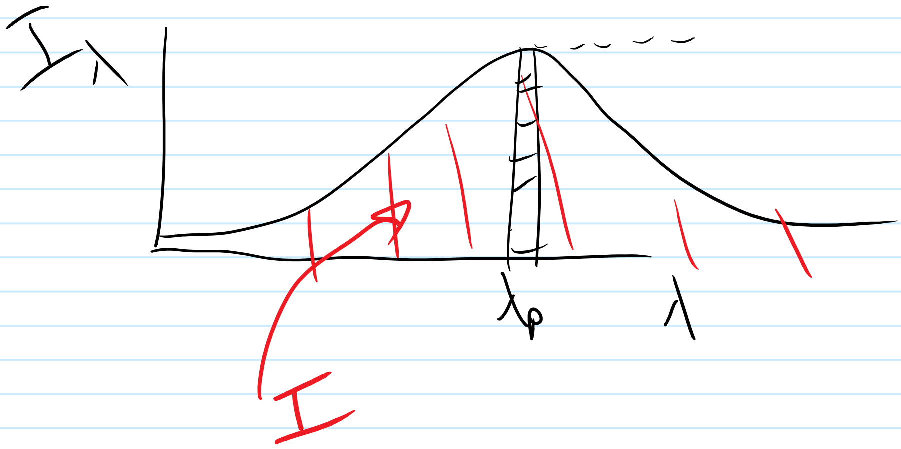
The spectrum of light emitted from a blackbody radiator has a peak intensity wavelength described (observed) by Wien's displacement LAW.

$$\lambda_p T = 2.898 \text{ mm K}$$

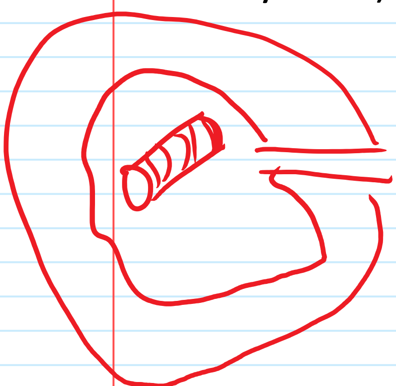
We can measure the intensity (Watts/m^2) over the entire spectrum by integrating the Intensity per unit wavelength over the entire spectrum

I $\qquad$ I_λ (different units than I)

$$I = \int_{all} I_\lambda \, d\lambda$$



Inside the cavity (the blackbody radiator cavity) we can describe the intensity as proportional to the Energy Density in a given region.   We'll relate the two (Energy Density, and Intensity later).



I can loosely think of that energy density as a tube of light on the move (I'll need to take 3Dim into account).

ν = frequency $\qquad \lambda$ = wavelength

$$U_\nu = \frac{\text{energy}}{\text{VOL}\ (\Delta\nu)} \qquad U_\lambda = \frac{\text{energy}}{\text{VOL}\ \Delta\lambda}$$

And $\qquad c = \nu\lambda$

The Wien law stated in terms of frequency (the peak frequency does not correspond to the peak wavelength directly).

$$\nu_p = 58.79\ \frac{\text{GHz}}{\text{K}} * T$$

In addition to the peak wavelength or frequency, we also have the total power output of the blackbody radiator

$$P_{total} = \underbrace{I}_{\text{total intensity } W/m^2} * \underbrace{A}_{\text{surface area}}$$

$$= \sigma \epsilon T^4 A$$

σ=56.71 nW/(m^2 K^4)

ϵ = Emissivity----ranges from 0 to 1
0 means perfect reflector
1 means perfect emitter
(diffuse white, or mirror shiny=0
black=1).

T is the absolute temperature IN KELVIN

A is the surface area in meters.

The T^4 law is the Stephan-Boltzman law, and the constant σ is known as the Stephan-Boltzman constant.

This law describes the total power emitted (rate of electromagnetic energy) spit out of an object with size A, temperature T----regardless of whether it is the person sitting next to you or a star 100 light years away.

We still need the spectrum----Wien, and Stephan-Boltzman give me the peak and total area.

A sidenote. The classical predictions indicated that at higher frequency there was more and more energy coming out of a blackbody radiator---and higher frequency light does have more energy, but there was no classical accounting for "HOW MANY PHOTONS"---the notion of photons did not yet exist. Physicists were not counting properly yet.

We shall learn to count now----

Breath deep---get ready---counting is really hard

6.2 The model

Planck suggests that we look at the cavity (for the moment) as a cube with sides labeled "L" --and we look on the inside, at light that may be emitted and reflected back and forth inside the cavity (remember we are only allowing it out the little hole---which won't matter later)

Try the PHET simulation (or your slinky)
https://phet.colorado.edu/sims/html/wave-on-a-string/latest/wave-on-a-string_en.html
https://phet.colorado.edu/sims/html/wave-on-a-string/latest/wave-on-a-string_en.html

Set to oscillate, Amplitude to ~0.1, no damping, highest tension. Fixed end. Try frequency ~0.41, then roughly double, triple, etc---watch what happens to amplitude.

At any of the frequencies not corresponding to a standing wave mode--the wave interferes itself away and is "not there" ==never builds up a big vibration.

Each length L is filled with waves that MUST correspond to standing wave mode, 1, or 2, 3,4,.........1000, 1001, ...100000, and so on

$$\begin{aligned} L &= 1\,\frac{\lambda_1}{2} \\ &= 2\,\frac{\lambda_2}{2} \\ &= 3\,\frac{\lambda_3}{2} \\ &= n\,\frac{\lambda_n}{2} \end{aligned}$$

For any mode, the length L is filled with integer half wave segments.

Since light has a really short wavelength, and object are typically big---we are going to have "n" being a high number.

We are going to assume that the system is in equilibrium--at least approximately.

If the frequency is any other frequency than standing wave mode---then the system /cavity --energy interferes itself away----Another way of saying, ---cavities can only hold onto energy in standing wave modes.

Standing wave modes

$$n_x \frac{\lambda_{n_x}}{2} = L_x$$

We can have y and z modes also independently (and also a second perpendicular polarization for light). It will be important that we COUNT all modes.

frequency
$$\nu_{n_x} = \frac{c}{\lambda_{n_x}} = \frac{n_x c}{2L}$$

These are the possible frequencies---corresponding to x modes. Each mode carries a weighting and carries energy. Knowledge about how energy is spread out over various modes (frequencies) ultimately gives us the spectrum (blackbody radiation spectrum).

We will start counting modes soon--and we will make a kind of change of variables. We will consider the object large (many many atoms/molecules)---so for light (with short wavelengths) the values of n will be big, and going from say mode 1 trillion to 1trillion plus 1---represents a very nearly continuous change in n (small relative changes).

$$n = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

(we will do an angle thing too soon---we get to define such things).

Coming important questions

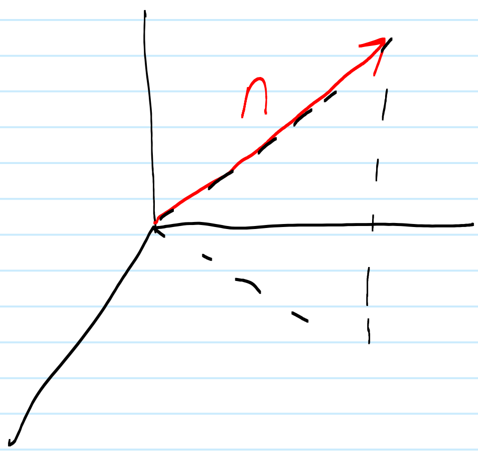
1) How many modes between n and (n + dn)
2) Density of modes (how many modes per real volume of space
3) How much energy per mode

With these answers we will be able to predict how much energy might be emitted at a given wavelength (frequency) from any object with given temperature.

Right now--counting standing wave modes.

$$n = \frac{2L}{c}\nu_n \qquad dn = \frac{2L}{c}d\nu_n$$

We want $\frac{\#\text{ modes}}{\text{volume}}$ and $\frac{\text{energy}}{\text{mode}}$ density



We will consider the # modes in a shell of radius n with thickness dn

Only keep 1 octant with positive n's

→ So will need $\frac{1}{8}$ the entire n shell.

A shell of "radius" n has an area in one octant of $4\pi n^2/8$ (the 1/8 is due to ---standing wave modes are positive--so only counting one octant in 3D space).

For a shell with thickness dn the volume of the shell will be

Vol Shell dn= $\frac{4\pi n^2 \, dn}{8}$

This is not a volume in "real space" but in "n" space--and being used to count how many modes there are when n_x n_y and n_z are varied throughout a shell as described.

We are counting.

Since, for light, any standing wave may have independent perpendicular polarizations ---we multiply by "2"

This volume is how many modes.

$$\# \text{Modes} = \frac{4\pi n^2 \; 2 \; dn}{8}$$

Now convert back to frequency since we have a relation

$$n = \frac{2L\nu_n}{c} \qquad dn = \frac{2L \, d\nu_n}{c}$$

$$\# \text{modes} = \frac{8\pi L^3 \nu^2 \, d\nu}{c^3}$$

This is the number of modes between ν and (ν+dν) ––
So…we have converted back to frequency (rather than "n") because we want to know about the spectrum--how much light at given frequencies.

If we divide both sides by L^3 (volume of the cavity) then ---being still completely general---we have the "density of modes".

$$\frac{\#\,\text{modes}}{\text{vol}} = \frac{8\pi\,\nu^2\,d\nu}{c^3}$$

If we can figure out how much energy is "stored" in each mode, then we can get the energy per volume---and it is energy on the move (light) ---and we know how fast. (we might get radiated power out of this).

We want---ENERGY PER MODE----
AND is there a statistical weighting for some modes over others?

The incorrect classical result had

Energy of a mode $= kT$

Boltzmann const $1.38\times10^{-23}\ \frac{J}{\text{Kelvin}}$

Which would give

$$\cancel{u_\nu\,d\nu = \frac{8\pi\,(kT)\,\nu^2 d\nu}{c^3}}$$

The classical result for Energy Density within a given frequency range ---has some incorrect results, such as at high frequency the energy density just gets bigger and bigger---OK-WE CANNOT HAVE INFINITE ENERGY DENSITY AT ANY FREQUENCY---BUT IS IT EVEN POSSIBLE TO HAVE MORE OF THE HIGH FREQUENCY STUFF (LIGHT)? WHY/WHY NOT.

The classical boo-boo--was called the UV catastrophe----

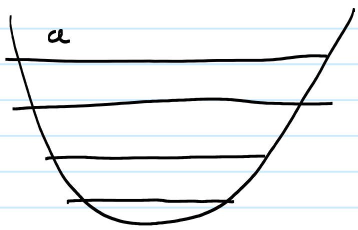
kT (Boltzmann const * absolute temp) does play a role . Recall k, is just a relation to the average kinetic energy per particle---per degree "Kelvin". Also recall that If you multiply k by Avagadro's number--you get the ideal gas constant (R---check it out---Joules/(Mol-K)

Where do the cavity modes get energy from — the Thermalized energy — The atoms or electrons — which are bound — and all bound systems are

Put answer here → $\boxed{?}$

The atoms/molecules---vibrate---are Harmonic Oscillators---or you can accept ---the things that made waves are little "vibrations" in the walls (the matter vibrates). All the different frequencies came from vibrations of matter.

OK so e^- and Atoms all do



S.H.O.

Classical says any energy is possible.

For classical simple harmonic oscillator ((1/2)kx^2) potential) the energy could be anything. But for systems like electrons orbiting atoms or molecules---THE ENERGY LEVELS ARE QUANTIZED AND DISCRETE.

Planck →

$E_n = n h\nu$ (really $(n+\frac{1}{2})h\nu$)

$n = 0, 1, 2 \ldots$

h = measured constant
$= 6.626 \times 10^{-34}$ J·s

The 1/2 does not really matter---all that matters is the quantization (which as validated for light with the photoelectric effect)

All bound systems contain discrete energy levels (best described using quantum mechanics). If the bound particle is low down in the potential well, all bottoms of wells look parabolic--and hence all bound systems are SIMPLE HARMONIC OSCILLATORS (at the bottom of the well).

We will use Planck's suggestion as the Energy for each mode.

$$E_n = n(h\nu)$$

There may be a broad range of frequencies around---we are only looking at ν right now, and want to know the average energy after accounting for all the different energy levels n----- we are reusing "n" differently now for oscillator energy levels (we were done counting standing wave modes).

We want $\overline{E_\nu}$ ave over all n

Statistics---mosh pits---Boltzman---collisions---etc.
All around you are gas molecules---(we can make a solid and consider phonons instead). The molecules move and undergo collisions on occasion. Sometimes a given molecule is hit by two others and may pick up (or lose) more energy. Even less often, the molecule may be hit by three others---and pick up even more energy. And so on.......Apparently there is a small likely hood that 10^{10} molecules will gang up and hit "one" at ~ the same time.

So---super high energies on one molecule ---not likely--UV Catastrophe--poof--gone. Statistics wins the day.

If you have enough particles, moving fast enough---maybe 4 collisions adding energy before losing much---maybe that is the norm (depending on the TEMPERATURE).

Enter ---Boltzman

The statistical probability of populating that "n"th mode of an oscillator (verified experimentally) and derived statistically is given by

$$\text{prob}_{n \text{ rel to gr}} = \frac{N_n}{N_0} = e^{-E_n/KT}$$

N_n = population of n
N_0 = pop ground

Math, counting and averaging.
Rearrange our Boltzman statistics equation

$$\text{total}_{\text{ALL states}} = \sum_{n \text{ all}} N_n = \sum N_0\, e^{-E_n/KT}$$

$$\text{Total states} = \sum_{n=0}^{\infty} N_0 e^{-nh\nu/KT}$$

rel prob to ground

and

$$E_{TOT} = \sum_{n=0}^{\infty} E_n (P_n N_0) \sim N_n$$

$$= \sum_{n=0}^{\infty} (nh\nu) N_0 e^{-nh\nu/KT}$$

$$\overline{E} = \frac{\sum E_n (P_n N_0)}{\text{total \# States}}$$

$$= \frac{\sum_{n=0}^{\infty} (nh\nu) N_0 e^{-nh\nu/KT}}{\sum N_0 e^{-nh\nu/KT}}$$

U

With the variable "u" substituted---(out front too) ---the series can be summed to yield the following result for the average energy.

$$\overline{E} = \frac{h\nu}{e^{+h\nu/KT} - 1}$$

To get the energy density we multiply by the density of states

to get energy density we multiply by the # oscillators per volume

$$\left(\frac{8\pi\nu^2}{c^3}\right) d\nu$$

$$U_\nu \, d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right]} d\nu$$

Let's check units just to see what this all means.

$$\underbrace{u_\nu \, d\nu}_{\text{Energy density}} = \frac{8\pi h \nu^3}{c^3} \frac{1}{\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]} d\nu$$

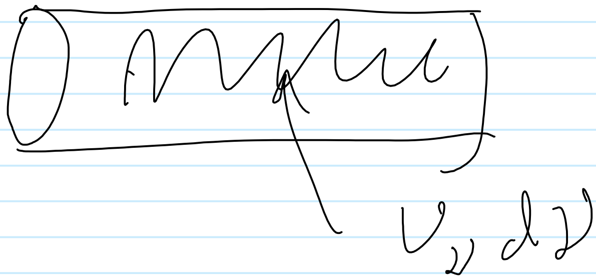
$$\frac{J\cdot\cancel{s}\cdot\frac{1}{\cancel{s^3}}}{m^3/\cancel{s^3}} \qquad \cancel{1/s}$$

In the cavity---heading back forth, up down, right, left, etc 2 polarizations. This is the Energy in a small volume-- and in that volume this energy is contained between frequency ν and $\nu + d\nu$. We can integrate and add up the total energy density (remember integration just means adding).

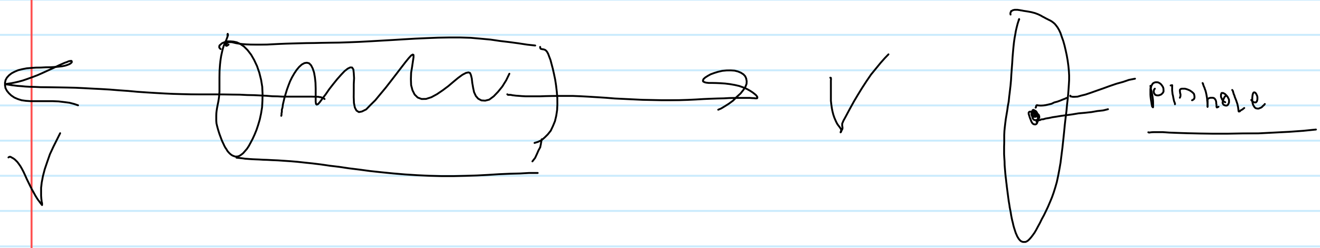
This result must be proportional to how much light is emitted from a small hole in the cavity---but we need to do a little geometry.

Pause--reflect (LOL--haha--reflect)---we have basically obtained (almost) the full spectrum by assuming quantization (Planck), and statistics (Boltzman)---oh and that vibrations are made by SHO.

We need to do a little more last geometric steps.



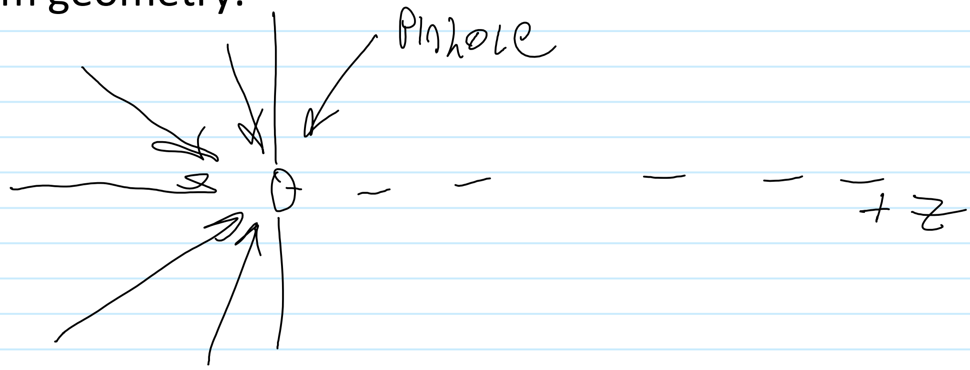
The energy density here is literally J/m^3 ---but that energy density (think of as water density) is on the move---and it is moving both ways.



Inside a cavity, energy reflects, comes back and forms standing wave. Motion is both ways. We want to know the rate at which the "stuff" (energy density) gets out a pinhole on the right. We must take only the right moving waves (half) and recognize that the rate at which "stuff" is moving out the hole is "c" the speed of light.

So far to get intensity (Joules/s)/m^2 coming out the pinhole, we must multiply result by c/2.

There is another……(skywalker---no….uh) another factor of 1/2 ….from geometry.



In a 3 dimensional world--light is carried through the pinhole if incident from angles +90 to -90 (in θ) and all the way around in φ. So, we must take the average "z" component of the "intensity" vector bringing light through the hole. Let us just call that vector I_o and we want to find the average z component of the I_o vector.

$$I_{oz} = I_o \cos\theta$$

$$\overline{I_{oz}} = \frac{\int_{Half} I_{oz}\, d\Omega}{(2\pi)}$$

Total solid angle half sphere

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

$$= \frac{I_o \int_0^{2\pi}\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta\, \cancel{d\phi}}{\cancel{2\pi}}$$

$$= \frac{1}{2}$$

So to turn our result into "intensity"---careful on my use of symbols here. We must net---multiply the energy density by c/4.

So

$$I_\nu\, d\nu = \frac{c}{4} U_\nu\, d\nu$$

$$U_\nu\, d\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{\left[\exp\left(\frac{h\nu}{KT}\right) - 1\right]} d\nu$$

$$\underbrace{I_\nu\, d\nu}_{\text{intensity}} = \frac{2\pi h \nu^3}{c^2} \frac{d\nu}{\exp\left(\frac{h\nu}{KT}\right) - 1}$$

To get TOTAL INTENSITY ONE MUST INTEGRATE OVER ALL FREQUENCIES.

Often this is cited with a factor of PI divided out to give (Watts/m^2)/steradian----denoting that on average light is emitted at some intensity per unit solid angle. (Instead of c/4---we would have c/2 ---c for the motion, 1/2 for the direction, and then $1/(2\pi)$ for averaging emission into any of the half sphere solid angle.

I have integrated over all the directions coming out of the pinhole (a factor of pi).

https://phet.colorado.edu/sims/html/blackbody-spectrum/latest/blackbody-spectrum_en.html

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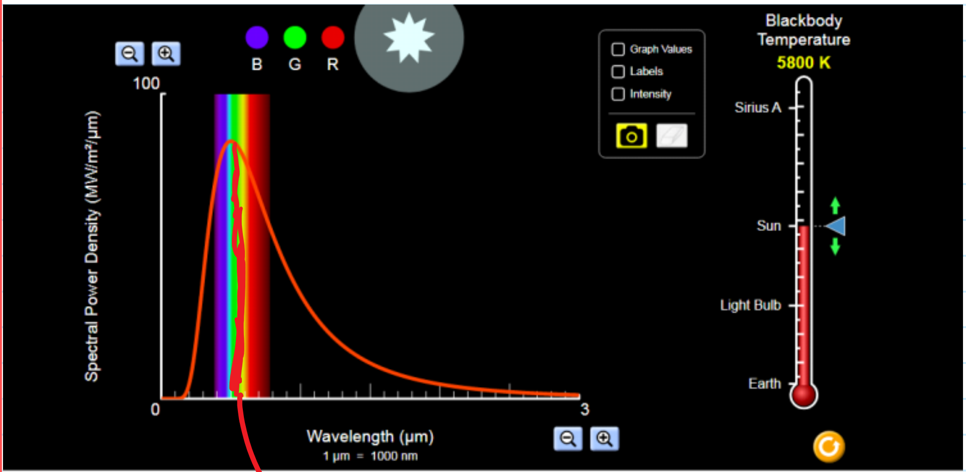
We can convert our equation to (Homework) $I_\lambda \, d\lambda$

We can convert to integrated total intensity--homework (and derive the Stephan-Boltzman law)

We can find the peak for wavelength or for frequency versions---HW.



Area here = $I_\lambda \, d\lambda$

As temp changes--peak height and wavelength changes, total area changes. Bigger T---more total area---much more. Bigger T--taller peak----Bigger T peak is at higher frequency (shorter wavelength).

Play with the simulation.

Be able to answer qualitative and quantitative questions.

We have derived ---the intensity of light (or spectral irradiance per unit frequency per steradian) emitted by a "hot object"....assuming ----quantization, and oscillators making waves---the rest is math. We made use of derivable Boltzman statistics.