MATH 3020 - **Differential Equations**
PRACTICE EXAM

Name——————————————————————

Write your solutions in a clear and precise manner.

Section 1
Answer all questions.

1. (**25pts**) Consider the differential equation

$$x' = (x-1)(5-x). \tag{1}$$

   a). Use appropriate method to solve for the solution satisfying the initial value $x(0) = 1.1$.

   b). Find the equilibrium point(s) of the differential equation (1).

   c). Plot the growth rate function and the phase line diagram (**label appropriately**).

   d). From your plot, what is the value of $\lim_{t\to\infty} x(t)$ for the following initial points:

      i). $x(0) = 0.9$,

      ii). $x(0) = 1.1$,

      iii). $x(0) = 5.01$.

   e). Describe the region where the curve is increasing, decreasing and constant.

2. (**10pts**) a). Show that $x(t) = Ae^{-t} + Be^{3t}$ is the general solution of the second order D.E.

$$x'' - 2x' - 3x = 0$$

   for any constant values A and B.

   b). Find two values of λ for which $x(t) = e^{\lambda t}$ is a solution of the D.E. $2x'' - 5x' - 3x = 0$.

3. (**20pts**) a). By hand, sketch the slope field for the D.E. $x' = -x + 2t$ (**use different values for** c **including positive, negative and zero value**).

   b). Describe the nullclines.

   c). Use the slope field to sketch the curve satisfying $x(1) = 2$.

4. (**20pts**) Find the general solution of the following D.E. using appropriate method. If using method for linear equations, clearly state the expression for the integrating factor, $\mu(t)$.

   a). $y' = \frac{2ty^2}{1+t^2}$, $y(0) = k$. Find interval of existence when $k > 0$, when $k < 0$ and $k = 0$.

   b). $x' = \left(a + \frac{b}{t}\right)x + t^2 e^{at}$. Consider finding solution when $b \neq 1$ and when $b = 1$.

   c). $x' = \frac{t+1}{\sqrt{t}}$, $x(1) = 4$.

Section 2
Answer any one question.

According to Newton's law of cooling (or, heating), the rate of change of the temperature T of an object is proportional to the difference $(T - T_0)$ of the object and the environmental temperature, T_0. **Use this law to answer any of the following question. Round all answers to 4 decimal places.**

5. (**25pts**) The body of a murder victim was discovered at 12:00 P.M. The medical examiner arrived at 12:30 P.M. and found the temperature of the body was 92.1^oF. The temperature of the room was 74^oF. Half hour later, in the same room, he took the body temperature again and found that it was 91.3^oF.

   a). Estimate the EXACT time of death (**rounded to the appropriate time**, e.g. 11:58 A.M.).

   b). What assumptions are being made?

   c). Evaluate $\lim_{t\to\infty} T(t)$. Is there any meaning for this value?

5. (**25pts**) A pan of water at 40^oC was put into a refrigerator at 10 A.M. If the temperature inside the refrigerator is -9^oC. Twenty minutes later the water was 27^oC.

   a). Estimate the temperature of the pan at 10:10 A.M.

   b). Estimate the time it takes the temperature of the water to reach 0^oC.

   c). Evaluate $\lim_{t\to\infty} T(t)$. Is there any meaning for this value?

5. (**25pts**) Suppose the temperature inside your winter home is 57^oF at 3:00 P.M. and your furnace then fails. If the outside temperature is 5^oF and you notice that by 11:00 P.M. the inside temperature is 46^oF.

   a). What is the temperature in your home the next morning at 8:30 A.M.?

   b). When (exact time) will the temperature falls to 38^oF?

   c). Evaluate $\lim_{t\to\infty} T(t)$. Is there any meaning for this value?

MATH 3020- Differential Equations
PRACTICE EXAM 2

Name————————————————————

Write your solutions in a clear and precise manner.
Answer all questions.
Good Luck.

1. (**15pts**) Consider the differential equation

$$x' = 2x - x^2. \tag{1}$$

   a). Plot the growth rate function and sketch the phase line diagram (**label properly**).

   b). Find the equilibrium points and classify them according to their stability.

   c). Describe the region where the curve is increasing, decreasing and constant.

   d). Sketch a time series plot satisfying $x(0) = 1/2$.

   e). Describe the long-term behavior of the solution in (d) (that is, find $\lim_{t\to\infty} x(t)$).

2. (**15pts**) Given the differential equation

$$x' = (x-1)(2x-h), \text{ where } h \text{ is a parameter.} \tag{2}$$

   a). Find the equilibria in terms of h.

   b). Determine the stability of the equilibria using Theorem 1.32 or any appropriate method.

   c). Construct a bifurcation diagram showing how equilibria depend on h (**label branches of curves as unstable or stable**)

   d). What is the value of h where there is significant change in the character of the equilibria?

3. (**15pts**) Given the differential equation

$$x' = f(t,x); \quad x(t_0) = x_0. \tag{3}$$

   a). What are the conditions that guarantee existence of a unique solution?

   b). Consider the initial value problem (IVP)

$$x' + \frac{1}{t-2}x = \frac{1}{t-5}; \quad x(3) = 2.$$

      i). using your conclusion in (a), does the IVP has a unique solution in some interval containing $t = 3$? **Explain**.

      ii). if your answer above in b(i) is YES, what is the largest interval where the solution exists?

iii). if your answer in b(i) is NO, does the IVP has any unique solution in some interval?

4. (**15pts**) Given the IVP differential equation

$$x'' - 6x' + 9 = 0; \quad x(0) = 1, \quad x'(0) = 0. \tag{4}$$

Find (**if it exists**)

a). the characteristic equation

b). the two independent solutions

c). the general solution

d). the unique solution

5. (**20pts**) Given the IVP differential equation

$$x'' + 4x' + 20 = 0; \quad x(0) = 2, \quad x'(0) = 0. \tag{5}$$

Find (**if it exists**)

a). the characteristic equation

b). the two independent solutions

c). the general solution

d). the unique solution

e). the unique solution in Phase-shift form $x(t) = e^{\alpha t}\left[R\cos\left(\beta t - \omega\right)\right]$

f). the amplitude of the solution

g). the phase-shift

h). the period of the solution

6. (**20pts**) **Solve** $x'' - 3x' - 4x = 2t^2 + t$; $x(0) = 1$, $x'(0) = 2$ and state

a). the homogeneous solution, $x_h(t)$

b). the particular solutions, $x_p(t)$

c). the general solution, $x(t)$

d). the unique solution

Form of source function $f(t)$	Trial form of particular solution $x_p(t)$
α	$x_p = A$
$\alpha e^{\beta t}$	$x_p = Ae^{\beta t}$
Polynomial of degree n	$x_p(t) = A_n t^n + A_{n-1}t^{n-1} + ... + A_1 t + A_0$
$\alpha \sin \omega t$; $\alpha \cos \omega t$;	$x_p(t) = A\alpha \sin \omega t + B\alpha \cos \omega t$;
$\alpha e^{rt} \alpha \sin \omega t$; $\alpha e^{rt} \alpha \cos \omega t$;	$x_p(t) = \alpha e^{rt} \alpha \sin \omega t + B\alpha e^{rt} \alpha \cos \omega t$

MATH 3020- Differential Equations
PRACTICE EXAM 3

Name————————————————————

Write your solutions in a clear and precise manner.
Answer all questions.
Good Luck.

1. (**20pts**) Consider the differential equation
$$t^2x^{''} + 4tx^{'} + 2x = 0. \tag{1}$$
   a). What type of differential equation is this?
   b). Find the general solution of the differential equation.
   c). Find the solution satisfying $x(1) = 1$; $x^{'}(1) = -2$.
   d). Sketch the solution and evaluate $\lim_{t\to\infty} x(t)$.

2. (**20pts**) By following the steps below, use the method of variation of parameter to solve the differential equation
$$x^{''} - 3x^{'} + 2x = e^{2t}, \quad x(0) = 1;\ x^{'}(0) = 1. \tag{2}$$
   a). Find two independent homogeneous solutions $x_1(t)$ and $x_2(t)$.
   b). What is the source function $f(t)$?
   c). Find the particular solution $x_p(t)$.
   d). Find the Wronskian function $W(t)$.
   e). Find the general solution $x(t)$.
   f). Find the solution, $x(t)$, satisfying the initial values given.

3. (**15pts**) One solution of the differential equation
$$x^{''} - \frac{t+2}{t}x^{'} + \frac{t+2}{t^2}x = 0 \tag{3}$$
is $x_1(t) = t$. Use the Reduction of order method to find a second independent solution.

4. (**15pts**) Use the definition to find the Laplace transform of the following.
   a). Give the expression/expression for the Laplace $\mathcal{L}[x(t)]$ of $x(t)$.
   b). Use this formula to find the Laplace transform of the following functions:
      a). $f(t) = \cos 5t$, HINT: Use the definition $\cos kt = \frac{1}{2}\left(e^{ikt} + e^{-ikt}\right)$
      b). $f(t) = e^{-5t}\sin 4t$, HINT: Use the definition $\sin kt = \frac{1}{2i}\left(e^{ikt} - e^{-ikt}\right)$
      c). $f(t) = H(t-3)$

5. (**10pts**) Find the inverse transform of the following.

a). $X(s) = \frac{6}{(s-1)^3}$,

b). $X(s) = e^{-7s}\frac{4s}{s^2+25}$,

c). $X(s) = \frac{3s}{(s-4)^2+7}$.

6. (**25pts**) Solve the following initial value problems using Laplace transform.

a). $x^{'} + 2x = \sin 3t, \quad x(0) = 1.$

b). $x^{''} - 2x^{'} - 3x = e^{2t}, \quad x(0) = 0; \; x^{'}(0) = 1.$

MATH 3020- Differential Equations
PRACTICE FINAL EXAM

Name————————————————————

Write your solutions in a clear and precise manner.
Answer all questions.

1. Find the general solution of the following D.E. using appropriate method.

   a). (**15pts**) Solve $\frac{dx}{dt} = (x-2)(4-x); \quad x(0) = 2.5.$

   b). (**15pts**) Solve $x' = \frac{x}{t} + te^{-t}; \qquad x(1) = 1.$

   c). (**10pts**) Solve $x' = \frac{t^2+1}{\sqrt[3]{t}}; \quad x(1) = 4.$

   d). (**15pts**) Use method of undetermined coefficient (state characteristic equation) to solve
   $$x'' + 2x' + 5x = 4t^2 + 2t; \ x(0) = 1; \quad x'(0) = 3.$$

   e). (**15pts**) Use method of variation of parameters to solve
   $$x'' - 2x' - 15x = e^{5t}; \ x(0) = 1; \quad x'(0) = 1.$$

   f). (**10pts**) Use Cauchy-Euler method to solve
   $$t^2x'' + 3tx' + x = 0; \ x(1) = 0; \quad x'(1) = 2.$$

   g). (**15pts**) One solution of the DE $x'' - \frac{1}{t}x' + \frac{1}{t^2}x = 0$ is $x_1(t) = t$. Use reduction of order method to find the other solution

2. (**10pts**) a). Sketch the slope field for the differential equation $x' = -x + 2t$.

   b). Use the slope field to sketch the solution satisfying $x(1) = 3$.

3. (**5pts**) Evaluate $\mathcal{L}^{-1}\left\{\frac{64}{(s-2)^5}\right\}$

4. (**10pts**) Use the **definition of Laplace transform** (not table) to find the transform of
   $$f(t) = e^{-4t}\cos 3t.$$
   HINT: Use the definition $\quad \cos kt = \frac{1}{2}\left(e^{ikt} + e^{-ikt}\right)$

5. (**15pts**) Solve the following initial value using Laplace transform.
   $$x'' - 2x' + 5x = 3e^{-t}, \quad x(0) = 0; \ x'(0) = 2.$$

6. (**15pts**) Consider the linear system of differential equations
   $$\begin{aligned} x' &= -y \\ y' &= 5x. \end{aligned} \qquad (1)$$

   a. Sketch the vector field $\mathbf{F}(x, y)$.

   b. Use method of elimination to find the general solution.